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Symplectic half-flat solvmanifolds

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Abstract We classify solvable Lie groups admitting left invariant symplectic half-flat structure. When the Lie group has a compact quotient by a lattice, we show that these structures provide solutions of supersymmetric equations of type IIA.

Keywords Symplectic half-flat structures · Solvable Lie algebras · Supersymmetric equations of type IIA

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1 Introduction

An SU(3) structure on a six-dimensional manifold defines a nondegenerate 2-form F, an almost complex structure J and a complex volume form Ψ. The SU(3) structure is *half-flat* if the 4-form $\sigma = F \wedge F$ and the 3-form $\rho = \Psi_+$ given by the real part of the complex volume form are both closed differential forms. Half-flat SU(3) structures are of interest

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in both differential geometry and physics, since they give rise to G_2 holonomy metrics in dimension 7 by solving a certain system of evolution equations [7,17].

Nilpotent Lie algebras with half-flat structures have been classified by Conti [4]. Schulte-Hengesbach [20] has classified direct sums of 2 three-dimensional Lie algebras admitting half-flat SU(3) structure, and the complete classification of decomposable half-flat Lie algebras is achieved by Freibert and Schulte-Hengesbach in [11]. Moreover, in the recent paper [12] they classify arbitrary indecomposable Lie algebras admitting a half-flat SU(3) structure, except for the solvable case with four-dimensional nilradical. We use here some of their results as it is explained below.

In the present paper we consider the case when F is itself closed, i.e. (F, Ψ) is a *symplectic half-flat* structure. It is known that a symplectic half-flat structure (F, Ψ) on a 6-manifold M defines, on the 7-manifold $M \times \mathbb{R}$, the 3-form $\varphi = F \wedge dt + \Psi_+$ which is a calibrated G_2 form in the sense of Harvey and Lawson [15]. Moreover, as we recall below, in the compact case this kind of structures are closely related to solutions of the supersymmetric equations of type IIA. Nilpotent Lie algebras having symplectic half-flat structure are classified by Conti and Tomassini [6]. In this paper, we classify the (non-nilpotent) solvable Lie algebras admitting symplectic half-flat structure and, as an application, solutions of such equations are given.

For unimodular solvable Lie algebras, the classification is the following:

Theorem 1.1 *An unimodular (non-Abelian) solvable Lie algebra* $\mathfrak{g}$ *has a symplectic half-flat structure if and only if it is isomorphic to one in the following list:*

$$\begin{aligned}
&\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1) = \left(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}\right);\\
&\mathfrak{g}_{5,1}\oplus\mathbb{R} = \left(0,0,0,0,e^{12},e^{13}\right);\\
&A_{5,7}^{-1,-1,1}\oplus\mathbb{R} = \left(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0\right);\\
&A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R} = \left(\alpha e^{15}+e^{25}, -e^{15}+\alpha e^{25}, -\alpha e^{35}+e^{45}, -e^{35}-\alpha e^{45}, 0, 0\right), \quad \alpha\geq 0;\\
&\mathfrak{g}_{6,N3} = \left(0,0,0,e^{12},e^{13},e^{23}\right);\\
&\mathfrak{g}_{6,38}^{0} = \left(e^{23}, -e^{36}, e^{26}, e^{26}-e^{56}, e^{36}+e^{46}, 0\right);\\
&\mathfrak{g}_{6,54}^{0,-1} = \left(e^{16}+e^{35}, -e^{26}+e^{45}, e^{36}, -e^{46}, 0, 0\right);\\
&\mathfrak{g}_{6,118}^{0,-1,-1} = \left(-e^{16}+e^{25}, -e^{15}-e^{26}, e^{36}-e^{45}, e^{35}+e^{46}, 0, 0\right).
\end{aligned}$$

It is worth noting that the corresponding solvable Lie groups admit a co-compact discrete subgroup (see Remarks 3.2 and 4.4 for details).

In the description of the Lie algebras, we are using the structure equations with respect to a basis $e^1, \ldots, e^6$ of the dual $\mathfrak{g}^*$. For instance, $\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1) = \left(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}\right)$ means that there is a basis $\{e^j\}_{j=1}^6$ satisfying $de^1 = 0$, $de^2 = -e^1\wedge e^3$, $de^3 = -e^1\wedge e^2$, $de^4 = 0$, $de^5 = -e^4\wedge e^6$ and $de^6 = -e^4\wedge e^5$; equivalently, the Lie bracket is given in terms of its dual basis $\{e_j\}_{j=1}^6$ by $[e_1, e_2] = e_3$, $[e_1, e_3] = e_2$, $[e_4, e_5] = e_6$ and $[e_4, e_6] = e_5$. For listing the Lie algebras we use the same labels as in the lists given in [3,21,24].

In Theorem 1.1 the Lie algebras $\mathfrak{g}_{5,1}\oplus\mathbb{R}$ and $\mathfrak{g}_{6,N3}$ are the only (non-Abelian) nilpotent Lie algebras having symplectic half-flat structure [6]. The Lie algebras $\mathfrak{g}_{6,N3}$, $\mathfrak{g}_{6,38}^{0}$, $\mathfrak{g}_{6,54}^{0,-1}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$ in Theorem 1.1 are indecomposable, whereas the first three Lie algebras and the family $A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R}$ are decomposable. The decomposable case $4\oplus 2$ is of special interest, because such Lie algebras (solvable or not) cannot admit symplectic half-flat structure (see Proposition 4.2 for details).

Regarding non-unimodular solvable Lie algebras, we have:

Theorem 1.2 *A non-unimodular solvable Lie algebra* $\mathfrak{g}$ *has a symplectic half-flat structure if and only if it is isomorphic to one in the following list:*

$$\begin{aligned}
A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1} &= \left(-\frac{1}{3}e^{16}+e^{23}, -\frac{2}{3}e^{26}, \frac{1}{3}e^{36}, e^{46}, -e^{56}, 0\right);\\
A_{6,54}^{2,1} &= \left(e^{16}+e^{35}, e^{26}+e^{45}, -e^{36}, -e^{46}, 2e^{56}, 0\right);\\
A_{6,70}^{\alpha,\frac{1}{2}\alpha} &= \left(\frac{\alpha}{2}e^{16}-e^{26}+e^{35}, e^{16}+\frac{\alpha}{2}e^{26}+e^{45}, -\frac{\alpha}{2}e^{36}-e^{46}, e^{36}-\frac{\alpha}{2}e^{46}, \alpha e^{56}, 0\right);\\
A_{6,71}^{-\frac{3}{2}} &= \left(\frac{3}{2}e^{16}+e^{25}, \frac{1}{2}e^{26}+e^{35}, -\frac{1}{2}e^{36}+e^{45}, -\frac{3}{2}e^{46}, e^{56}, 0\right);\\
N_{6,13}^{0,-2,0,2} &= \left(-2e^{16}, 2e^{26}, e^{36}-e^{45}, e^{35}+e^{46}, 0, 0\right).
\end{aligned}$$

Therefore, $\mathfrak{g}$ *is indecomposable.*

In the proof of Theorem 1.1 we use the classification given in [18] of six-dimensional unimodular solvable Lie algebras admitting a symplectic form. That list is given using as starting point the original classification due to Mubarakzyanov [19]. To prove Theorem 1.2 we use the classification of Turkowski [24] (or Freibert's and Schulte-Hengesbach's refinement [12] of Mubarakzyanov [19] classification) of solvable Lie algebras with four-dimensional (or five-dimensional, respectively) nilradical, and firstly we show which of those Lie algebras have a symplectic form (see Propositions 3.3 and 3.4). Moreover, for each Lie algebra given in the theorems above we show an explicit symplectic half-flat structure (F, Ψ). The last assertion of Theorem 1.1, i.e. that the Lie group associated to each Lie algebra admits a co-compact discrete subgroup, follows from [3,9,13,23,25].

The paper is structured as follows. In Sect. 2 we review general facts about SU(3) structures and explain the method that we follow for studying the existence of symplectic half-flat structure on solvable Lie algebras. In Sect. 3 we investigate the case when the solvable Lie algebra is indecomposable and, in particular, when the Lie algebra is non-unimodular and with nilradical of dimensions 4 or 5. Here we must notice that according to [19], a solvable Lie algebra of dimension 6 with nilradical of dimension lower than 4 is decomposable or nilpotent. The decomposable case is considered in Sect. 4. As a consequence of the results proved in Sect. 4 we conclude that the decomposable solvable Lie algebras having symplectic half-flat structure are unimodular.

Finally, as an application of Theorem 1.1, in Sect. 5 we provide compact solutions to certain supersymmetric equations of type IIA. Supersymmetric flux vacua with constant intermediate SU(2) structure were introduced by Andriot [1], and it is showed in [10] that they are closely related to the existence of special classes of half-flat structures on the internal manifold. In particular, solutions of the SUSY equations of type IIA possess a symplectic half-flat structure. In Proposition 5.3 we prove that any six-dimensional compact solvmanifold admitting an invariant symplectic half-flat structure also admits a solution of the SUSY equations of type IIA. This provides the complete list of compact solvmanifolds admitting invariant solutions of such supersymmetric equations.

2 Symplectic half-flat structures

In this section we recall some well-known facts about SU(3) structures and consider several obstructions to the existence of symplectic half-flat structures. We will follow ideas given in [4,5,11].

An SU(3) structure on a six-dimensional manifold M is an SU(3) reduction of the frame bundle of M. We consider the characterization of SU(3) structures given in [16], i.e. in terms of certain stable forms which satisfy some additional compatibility conditions.

A 3-form ρ on a six-dimensional oriented vector space (V, ν) is *stable* if its orbit under the action of the group GL(V) is open. Let $\kappa : \Lambda^5 V^* \longrightarrow V \otimes \Lambda^6 V^*$ be the isomorphism with $\kappa(\eta) = X \otimes \nu$ such that $\iota_X \nu = \eta$, and let $K_\rho : V \longrightarrow V \otimes \Lambda^6 V^*$ be given by $K_\rho(X) = \kappa(\iota_X \rho \wedge \rho)$. In terms of the invariant $\lambda(\rho) = \frac{1}{6}\mathrm{tr}(K_\rho^2)$, the stability of ρ is equivalent to the open condition $\lambda(\rho) \neq 0$. Moreover, $\lambda(\rho)$ enables us to construct a volume form $\phi(\rho) := \sqrt{|\lambda(\rho)|} \in \Lambda^6 V^*$.

The endomorphism $J_\rho := \frac{1}{\phi(\rho)} K_\rho$ gives rise to an almost complex structure if $\lambda(\rho) < 0$. The action of J_ρ^* on 1-forms is given by the formula

$$J_\rho^* \alpha(X) \phi(\rho) = \alpha \wedge \iota_X \rho \wedge \rho. \tag{1}$$

The characterization of SU(3) structures requires the existence of a 2-form F which is *stable*, i.e. $\phi(F) := F^3 \neq 0$, such that the pair (F, ρ) is *compatible*, in the sense that

$$F \wedge \rho = 0,$$

and *normalized*, i.e.

$$\phi(\rho) = 2\phi(F).$$

Such a pair (F, ρ) induces a pseudo Euclidean metric $g(\cdot, \cdot) = F(J_\rho \cdot, \cdot)$ which satisfies on 1-forms the identity

$$\alpha \wedge J_\rho^* \beta \wedge F^2 = \frac{1}{2} g(\alpha, \beta) F^3, \quad \alpha, \beta \in V^*. \tag{2}$$

An SU(3) structure on V is a pair of compatible and normalized stable forms $(F, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ with $\lambda(\rho) < 0$ inducing a positive-definite metric. If in addition $V = \mathfrak{g}$ is a Lie algebra then a *symplectic half-flat structure* on $\mathfrak{g}$ is an SU(3) structure (F, ρ) such that $dF = 0$ and $d\rho = 0$, where d denotes the Chevalley-Eilenberg differential of $\mathfrak{g}$.

From now on, we denote by $Z^k(\mathfrak{g})$ the space of closed k-forms on $\mathfrak{g}$, by $\mathcal{S}(\mathfrak{g}) = \{F \in Z^2(\mathfrak{g}) \mid F^3 \neq 0\}$ the space of symplectic forms on $\mathfrak{g}$, and by Ann(ρ) the annihilator of ρ in the exterior algebra $\Lambda^* \mathfrak{g}^*$. In [11] the authors give a useful simple obstruction to the existence of half-flat structures on Lie algebras based on Eq. (2). In the presence of a compatible symplectic form, such obstruction reads as:

Proposition 2.1 *Let us fix a volume element ν on $\mathfrak{g}$ and let $\rho \in Z^3(\mathfrak{g})$. Let us define $\tilde{J}_\rho^*$ by imitating (1) but with respect to the volume element ν, that is,*

$$(\tilde{J}_\rho^* \alpha)(X)\, \nu = \alpha \wedge \iota_X \rho \wedge \rho, \quad X \in \mathfrak{g},$$

and let $\tilde{J}_\rho \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ be the endomorphism of $\mathfrak{g}$ given by $\alpha(\tilde{J}_\rho X) = -(\tilde{J}_\rho^* \alpha)(X)$, for any $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. Then:

(i) *If there is some $\alpha \in \mathfrak{g}^*$ such that*

$$\alpha \wedge \tilde{J}_\rho^* \alpha \wedge F^2 = 0$$

for any $F \in \mathcal{S}(\mathfrak{g}) \cap \mathrm{Ann}(\rho)$*, then* $\mathfrak{g}$ *does not admit any symplectic half-flat structure.*

(ii) *If there are some* $X, Y \in \mathfrak{g}$ *such that*

$$F(\tilde{J}_\rho(X), X)\, F(\tilde{J}_\rho(Y), Y) \leq 0$$

for any $F \in \mathcal{S}(\mathfrak{g}) \cap \mathrm{Ann}(\rho)$*, then* $\mathfrak{g}$ *does not admit any symplectic half-flat structure.*

Proof Notice that $\tilde{J}_\rho$ is proportional to the almost complex structure J_ρ. Now, from (2) it follows that in case (i) the induced metrics g are degenerate, and in case (ii) we get a contradiction with the positive-definiteness of g. □

In [5] it is proved the following restriction to the existence of a calibrated G_2 form on a Lie algebra $\mathfrak{h}$: if there is a nonzero vector $X \in \mathfrak{h}$ such that $(i_X\phi)^3 = 0$ for all $\phi \in Z^3(\mathfrak{h}^*)$, then $\mathfrak{h}$ does not admit calibrated G_2 structures. It is known [8] that a symplectic half-flat structure on the Lie algebra $\mathfrak{g}$ induces the calibrated G_2 form $\varphi = F \wedge dt + \Psi_+$ on the Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$. These facts imply the following:

Proposition 2.2 *If there exists* $X \in \mathfrak{g} \oplus \mathbb{R}$ *such that*

$$(i_X\varphi)^3 = 0 \quad \textit{for all } \varphi \in Z^3((\mathfrak{g} \oplus \mathbb{R})^*),$$

then $\mathfrak{g}$ *does not admit symplectic half-flat structure.*

In this paper we study the existence of symplectic half-flat structures on six-dimensional solvable Lie algebras. In the decomposable cases $3 \oplus 3$, $4 \oplus 2$ and $5 \oplus 1$ our starting point is the half-flat classification obtained in [11,20] and, after finding which of them admit symplectic forms, we apply the above proposition to classify the Lie algebras having symplectic half-flat structure. We remark that the results obtained in the decomposable cases are completely general and not constrained to the solvable type.

On the other hand, in the case of indecomposable unimodular solvable Lie algebras, we start with the symplectic classification given in [18] and then we search for a compatible half-flat structure (see Proposition 3.1). For indecomposable non-unimodular solvable Lie algebras with nilradical of dimension 4, we use the list of [24] and we show which of them have a symplectic form (see Proposition 3.4). However, in the case of indecomposable non-unimodular solvable Lie algebras, with five-dimensional nilradical, we begin with the classification given in [12] of indecomposable solvable Lie algebras having a half-flat structure (see Proposition 3.3). The next step is then to apply the above propositions which are obstructions to the existence of symplectic half-flat structures.

In the cases of existence of symplectic half-flat structure we will only provide a particular example (F, ρ) written in standard form $F = f^{12} + f^{34} + f^{56}$ and $\rho = Re((f^1 + if^2)(f^3 + if^4)(f^5 + if^6))$ with respect to some basis $\{f^1, \ldots, f^6\}$ of $\mathfrak{g}^*$. Notice that this basis is orthonormal for the underlying metric and the almost complex structure J is given by $J^*f^1 = -f^2$, $J^*f^3 = -f^4$ and $J^*f^5 = -f^6$.

3 The indecomposable case

In this section we prove Theorems 1.1 and 1.2 in the case of indecomposable solvable Lie algebras of dimension 6.

In the next proposition we focus on unimodular solvable Lie algebras which are not nilpotent, because it is proved in [6] that $\mathfrak{g}_{6,N3}$ is the only indecomposable nilpotent Lie algebra admitting symplectic half-flat structure. For listing the unimodular solvable Lie algebras, we use the notation given in [3] (see Appendix).

Proposition 3.1 *The only six-dimensional indecomposable unimodular non-nilpotent solvable Lie algebras admitting symplectic half-flat structures are* $\mathfrak{g}_{6,38}^{0}$, $\mathfrak{g}_{6,54}^{0,-1}$ *and* $\mathfrak{g}_{6,118}^{0,-1,-1}$.

Proof By [18, Theorem 2] the indecomposable unimodular non-nilpotent solvable Lie algebras admitting a symplectic form are those appearing in Table 1 of the Appendix.

Notice that the Lie algebra $\mathfrak{g}_{6,54}^{0,-1}$ has a symplectic half-flat structure by [22]. Moreover, explicit symplectic half-flat structures on $\mathfrak{g}_{6,38}^{0}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$ are given in Table 1.

Next we show in some detail how Proposition 2.1 is used for the remaining Lie algebras $\mathfrak{g}$ in the list. In all cases we consider on $\mathfrak{g}$ the volume element given by $\nu = e^{123456}$, where $\{e^1, \dots, e^6\}$ is the basis of $\mathfrak{g}^*$ in Table 1.

On the Lie algebra $\mathfrak{g} = \mathfrak{g}_{6,3}^{0,-1}$, any pair (F, ρ) with $F \in Z^2(\mathfrak{g})$ and $\rho \in Z^3(\mathfrak{g})$ is given by

$$
\begin{aligned}
F &= b_1 e^{16} + b_2 e^{23} + b_3 e^{26} + b_4 e^{36} + b_5 e^{45} + b_6 e^{46} + b_7 e^{56},\\
\rho &= a_1 e^{123} + a_2 e^{126} + a_3 e^{136} + a_4 e^{146} + a_5 e^{156} + a_6 e^{236} + a_7 e^{246}\\
&\quad + a_8 e^{256} + a_9 e^{345} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}.
\end{aligned}
$$

The stability of F implies that $b_1, b_2, b_5 \neq 0$, so we can take without loss of generality $b_5 = 1$. From the compatibility condition $F \wedge \rho = 0$ we get $a_1 = a_2 = a_4 = a_5 = 0$, $a_3 = a_9 b_1$ and $a_6 = a_9 b_3 - a_{12} b_2$. Now, Proposition 2.1 (i) is satisfied for the 1-form $\alpha = e^6$ and consequently $\mathfrak{g}$ does not admit symplectic half-flat structure.

For the Lie algebra $\mathfrak{g} = \mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$, any pair $(F, \rho) \in Z^2(\mathfrak{g}) \times Z^3(\mathfrak{g})$ is given by

$$
\begin{aligned}
F &= b_1 e^{13} + b_2 \left(-\frac{1}{2} e^{16} + e^{23}\right) + b_3 e^{24} + b_4 e^{26} + b_5 e^{36} + b_6 e^{46} + b_7 e^{56},\\
\rho &= a_1 e^{126} + a_2 e^{135} + a_3 e^{136} + a_4 \left(\frac{1}{2} e^{146} - e^{234}\right) + a_5 \left(\frac{1}{2} e^{156} + e^{235}\right) + a_6 e^{236}\\
&\quad + a_7 e^{245} + a_8 e^{246} + a_9 e^{256} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}.
\end{aligned}
$$

The form F is stable if and only if $b_1, b_3, b_7 \neq 0$, so we can suppose without loss of generality that $b_1 = 1$. When imposing $F \wedge \rho = 0$ we find that $a_7 = -a_2 b_3$, $a_8 = -a_3 b_3$, $a_9 = a_5 b_2 + a_2 b_4$, $a_5 = a_2 b_2$, $a_{12} = a_2 b_6$ and $a_4 = \frac{-a_{11} b_3 - a_2 b_3 b_5}{b_7}$. A direct calculation shows that

$$
\tilde{J}_\rho e_1 = -\frac{b_3(a_{11} b_2 + b_2 b_5 + (a_2 - 1) a_3 b_7)}{b_7} e_1 + \frac{a_2 b_3 (a_{11} + b_5)}{b_7} e_2 + 2 a_1 a_2\, e_4,
$$

which implies that $F(\tilde{J}_\rho e_1, e_1) = 0$ and so Proposition 2.1 (ii) is satisfied for $X = Y = e_1$.

On the Lie algebra $\mathfrak{g} = \mathfrak{g}_{6,70}^{0,0}$, any pair $(F, \rho) \in Z^2(\mathfrak{g}) \times Z^3(\mathfrak{g})$ is given by

$$
\begin{aligned}
F &= b_1 \left(e^{13} + e^{24}\right) + b_2 \left(e^{16} + e^{45}\right) + b_3 \left(e^{26} - e^{35}\right) + b_4 e^{34} + b_5 e^{36} + b_6 e^{46} + b_7 e^{56},\\
\rho &= a_1 e^{125} + a_2 \left(e^{135} + e^{245}\right) + a_3 e^{136} + a_4 \left(e^{145} - e^{235}\right) + a_5 \left(e^{146} + e^{236}\right)\\
&\quad + a_6 e^{156} + a_7 e^{246} + a_8 e^{256} + a_9 e^{345} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}.
\end{aligned}
$$

The form F is stable if and only if $b_1, b_7 \neq 0$, so we consider $b_1 = 1$. From the condition $F \wedge \rho = 0$ we get that $a_2 = \frac{a_1 b_4}{2}$, $a_3 = -a_7$, $a_8 = a_2 b_3 - a_1 b_5 + a_4 b_2$, $a_6 = a_2 b_2 + a_1 b_6$, $a_{11} = (a_4 + a_5) b_2 + (a_7 - a_9) b_3 + a_8 b_4 + a_2 b_5$ and $a_{12} = (a_9 - a_3) b_2 - (a_4 + a_5) b_3 - a_6 b_4 + a_2 b_6$. We find that $\tilde{J}_\rho e_1 = \alpha_1 e_1 + \dots + \alpha_4 e_4$ and $\tilde{J}_\rho e_2 = \beta_1 e_1 + \dots + \beta_4 e_4$ with $\beta_4 = -\alpha_3 = -2 a_1 (a_4 + a_5)$. This implies that

$$
F(\tilde{J}_\rho e_1, e_1) = F(\alpha_1 e_1 + \dots + \alpha_4 e_4, e_1) = -\alpha_3 = -2 a_1 (a_4 + a_5)
$$

and

$$F(\tilde{J}_\rho e_2, e_2) = F(\beta_1 e_1 + \cdots + \beta_4 e_4, e_2) = -\beta_4 = 2a_1(a_4 + a_5).$$

Therefore, Proposition 2.1 (ii) is satisfied for $X = e_1$ and $Y = e_2$, and $\mathfrak{g}_{6,70}^{0,0}$ does not admit symplectic half-flat structure.

For the Lie algebras $\mathfrak{g}_{6,10}^{0,0}$, $\mathfrak{g}_{6,18}^{-1,-1}$, $\mathfrak{g}_{6,21}^{0}$ and $\mathfrak{g}_{6,36}^{0,0}$ one can prove that Proposition 2.1 (i) is satisfied for the 1-form $\alpha = e^6$ and consequently they do not admit symplectic half-flat structure.

For the Lie algebras $\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$ and $\mathfrak{g}_{6,78}$ a similar argument proves that $F(\tilde{J}_\rho e_1, e_1) = 0$, for $\mathfrak{g}_{6,15}^{-1}$ one has that $F(\tilde{J}_\rho e_4, e_4) = 0$ and for $\mathfrak{n}_{6,84}^{\pm 1}$ we have $F(\tilde{J}_\rho e_2, e_2) = 0$. Thus, by Proposition 2.1 (ii) these Lie algebras do not admit symplectic half-flat structure. □

Remark 3.2 The solvable Lie group corresponding to $\mathfrak{g}_{6,38}^{0}$ admits a lattice by [3, Proposition 8.3.3]. For $\mathfrak{g}_{6,54}^{0,-1}$, it is shown in [9] that the corresponding simply connected Lie group admits a compact quotient. Finally, the solvable Lie group corresponding to $\mathfrak{g}_{6,118}^{0,-1,-1}$ admits a lattice by [25].

In the following we study non-unimodular Lie algebras. A solvable Lie algebra of dimension 6 with nilradical of dimension lower than 4 is decomposable or nilpotent [19]. So, we are left to study Lie algebras with nilradical of dimensions 4 and 5.

To prove next proposition we use the list of Freibert and Schulte-Hengesbach [12] of the solvable Lie algebras, with five-dimensional nilradical, having a half-flat structure. There the authors use the corrected version, due to Shabanskaya [21], of the original classification by Mubarakzyanov [19] of solvable Lie algebras of dimension 6 with five-dimensional nilradical. That list given in [21] contains 52 Lie algebras and 70 families depending at least of one-parameter: 44 one-parameter families, 22 two-parameter families, 3 three-parameter families and 1 four-parameter family.

Proposition 3.3 *The only six-dimensional non-unimodular solvable Lie algebras with five-dimensional nilradical admitting symplectic half-flat structures are* $A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$, $A_{6,54}^{2,1}$, $A_{6,70}^{\alpha,\frac{\alpha}{2}}$ $(\alpha \neq 0)$ *and* $A_{6,71}^{-\frac{3}{2}}$.

Proof From the list given in [12] one can check that the unique non-unimodular solvable Lie algebras with five-dimensional nilradical admitting a symplectic form are those given in Table 2 of the Appendix. Next we use Proposition 2.2 to find which of them admit in addition a symplectic half-flat structure.

Let $\mathfrak{g} = A_{6,39}^{\frac{3}{2},-\frac{3}{2}}$ and consider the seven-dimensional Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$. Let us consider an arbitrary $\varphi \in Z^3(\mathfrak{h}^*)$. Then,

$$\begin{aligned}\varphi = {} & a_1 e^{146} + a_2 e^{156} + a_3\left(e^{135} + 2e^{236}\right) + a_4 e^{245} - a_5\left(e^{145} - e^{246}\right) + a_6 e^{256} \\ & + a_7\left(2e^{157} + e^{267}\right) - a_8\left(e^{136} - e^{345}\right) + a_9 e^{346} + a_{10} e^{347} + a_{11} e^{356} + a_{12} e^{367} \\ & + e_{13} e^{456} - a_{14}\left(e^{167} - 2e^{457}\right) + a_{15} e^{467} + a_{16} e^{567}.\end{aligned}$$

Put $\nu = i_{e_1}\varphi$. Thus,

$$\nu = a_1 e^{46} + a_2 e^{56} + a_3 e^{35} - a_5 e^{45} + 2a_7 e^{57} - a_8 e^{36} - a_{14} e^{67}$$

is a degenerate 2-form and Proposition 2.2 is fulfilled with the vector e_1, so $\mathfrak{g}$ does not admit symplectic half-flat structures.

Similarly, for $\mathfrak{g} = A_{6,39}^{1,-1}$ one has that Proposition 2.2 is satisfied for the vector e_2.

Now, if $\mathfrak{g}$ is the Lie algebra $A_{6,42}^{-1}$ then Proposition 2.2 is satisfied for e_2.

For the Lie algebras $\mathfrak{g} = A_{6,51}^{\pm 1}$, in both cases Proposition 2.2 can be checked with the vector e_3.

Now, if $\mathfrak{g}$ is the Lie algebra $A_{6,54}^{-1,-2}$ then Proposition 2.2 is satisfied for e_1.

We study now the family of Lie algebras $A_{6,54}^{\alpha,\alpha-1}$ with $0 < \alpha \leq 2$. If $\alpha = 0$ then it is unimodular, and the case $\alpha = 2$ will be studied at the end of the proof. On this family Proposition 2.2 is satisfied for the vector e_1.

For $A_{6,56}^{1}$ one gets that Proposition 2.2 is satisfied for the vector e_1, hence $A_{6,56}^{1}$ does not admit symplectic half-flat structure.

The Lie algebra $A_{6,65}^{1,2}$ does not admit a symplectic half-flat structure since Proposition 2.2 is satisfied for the vector e_2.

For the Lie algebra $A_{6,76}^{-3}$ we have Proposition 2.2 fulfilled for the vector e_1.

On the Lie algebra $A_{6,82}^{2,5,9}$, vector e_1 satisfies Proposition 2.2.

The Lie algebra $A_{6,94}^{-3}$ has no symplectic half-flat structure because Proposition 2.2 is satisfied for the vector e_1.

On the Lie algebra $A_{6,94}^{-\frac{5}{3}}$, vector e_1 satisfies Proposition 2.2.

The Lie algebra $A_{6,94}^{-1}$ has no symplectic half-flat structure because Proposition 2.2 is satisfied for the vector e_1.

Finally, explicit symplectic half-flat structures on $A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$, $A_{6,54}^{2,1}$, $A_{6,70}^{\alpha,\frac{\alpha}{2}}$ ($\alpha \neq 0$) and $A_{6,71}^{-\frac{3}{2}}$ are given in Table 2 of the Appendix. □

To complete the proof of Theorem 1.2 in the indecomposable case, it remains to study the solvable Lie algebras with four-dimensional nilradical. For this, we use the list of [24] that contains 12 Lie algebras and 31 families depending at least of one-parameter. Indeed, there are 14 one-parameter families, 10 two-parameter families, 4 three-parameter families and 3 four-parameter.

Proposition 3.4 $N_{6,13}^{0,-2,0,2}$ *is the only six-dimensional non-unimodular solvable Lie algebra with four-dimensional nilradical admitting a symplectic half-flat structure.*

Proof Starting from the list of Turkowski [24] of non-unimodular solvable Lie algebras with four-dimensional nilradical, we reduce our attention to those admitting a symplectic form, which we list in Table 3 of the Appendix.

An explicit symplectic half-flat structure (F, ρ) on the Lie algebra $N_{6,13}^{0,-2,0,2}$ is given in Table 3. For any of the remaining Lie algebras we use Propositions 2.1 and 2.2 to prove non-existence of symplectic half-flat structures. Indeed, for the families $N_{6,2}^{-1,\beta,-\beta}$, $N_{6,7}^{0,\beta,0}$ ($\beta \neq 0$) and $N_{6,13}^{\alpha,\beta,-\alpha,-\beta}$, with $\alpha^2 + \beta^2 \neq 0$ and $\beta \neq \pm 2$, as well as for the Lie algebra $N_{6,17}^{0}$ the hypothesis of Proposition 2.2 is satisfied with $X = e_3$. The family of Lie algebras $N_{6,2}^{0,-1,\gamma}$ satisfies Proposition 2.2 with the vector $X = e_2$ for all γ. Almost all of the remaining Lie algebras and parameter families appearing in Table 3 satisfy the hypothesis of Proposition 2.2 with the vector $X = e_1$. However, for the Lie algebra $N_{6,28}$ and the families $N_{6,1}^{\alpha,\beta,-\alpha,-\beta}$, $N_{6,1}^{\alpha,\beta,0,-1}$ and $N_{6,1}^{\alpha,\beta,-1,0}$, where Proposition 2.2 is not enough, we use Proposition 2.1 to assert that they do not admit symplectic half-flat structure. For this, we proceed as in the proof of Proposition 3.1 for $A^{0,0}_{6,70}$. We see that the Lie algebra $N_{6,28}$ satisfies Proposition 2.1 (ii) for $X = e_5$ and $Y = e_6$. For the family $N^{\alpha,\beta,-\alpha,-\beta}_{6,1}$ the obstruction (i) in Proposition 2.1 is satisfied for $(2+\sqrt{3})e^5 + e^6$. For $N^{\alpha,\beta,0,-1}_{6,1}$ Proposition 2.1 (i) is satisfied for the 1-form $(1+\sqrt{3})e^5 + e^6$ or $(1+\sqrt{3})e^5 - e^6$ depending on the sign of β. Finally, for the family $N^{\alpha,\beta,-1,0}_{6,1}$ the obstruction (i) in Proposition 2.1 is satisfied for the 1-form $(1+\sqrt{3})e^5 + 2e^6$ or $(1+\sqrt{3})e^5 - 2e^6$ depending on the sign of α, which completes the proof. □

4 The decomposable case

In this section we consider all six-dimensional Lie algebras (unimodular and non-unimodular) of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Our starting point is the half-flat classification given in [20] for the 3⊕3 case and in [11] for the 4⊕2 and 5⊕1 cases.

As in the previous section, we will suppose that $\mathfrak{g}$ is non-nilpotent, because it is proved in [6] that the only decomposable nilpotent Lie algebras having symplectic half-flat structure are $\mathfrak{g}_{5,1} \oplus \mathbb{R}$ and the Abelian Lie algebra. Therefore, Theorem 1.1 follows as a consequence of [6] and Propositions 3.1, 4.1, 4.2 and 4.3.

In the next result we consider the six-dimensional Lie algebras of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 = 3$.

Proposition 4.1 *The only* 3⊕3 *non-nilpotent Lie algebra which admits a symplectic half-flat structure is* $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$.

Proof Starting from the classification in [20], one can check that the 3 ⊕ 3 non-nilpotent Lie algebras admitting both half-flat structure and symplectic form are those listed in Table 4 of the Appendix. (Notice that in particular the only 3 ⊕ 3 non-unimodular solvable Lie algebras having both symplectic and half-flat structure are $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.)

A direct calculation shows that the Lie algebra $\mathfrak{e}(2) \oplus \mathfrak{e}(2)$ satisfies Proposition 2.1 (ii) for $X = e_1$ and $Y = e_2$.

For the Lie algebra $\mathfrak{e}(2) \oplus \mathbb{R}^3$ we have that $F(\tilde{J}_\rho e_2, e_2) = 0$. Let $\mathfrak{g} = \mathfrak{e}(1,1) \oplus \mathbb{R}^3$, since $Z^2(\mathfrak{g}) = Z^2(\mathfrak{e}(2) \oplus \mathbb{R}^3)$ and $Z^3(\mathfrak{g}) = Z^3(\mathfrak{e}(2) \oplus \mathbb{R}^3)$ the same argument as in the previous case is valid, so $F(\tilde{J}_\rho e_2, e_2) = 0$.

The Lie algebra $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$ satisfies Proposition 2.1 (ii) for $X = e_2$ and $Y = e_3$.

For $\mathfrak{e}(2) \oplus \mathfrak{h}$ and $\mathfrak{e}(1,1) \oplus \mathfrak{h}$ one has that $F(\tilde{J}_\rho e_6, e_6) = 0$.

The Lie algebras $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ both satisfy the obstruction (i) in Proposition 2.1 for $\alpha = e^4$.

Finally, it is proved in [22] that the Lie algebra $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ has a symplectic half-flat structure (see Table 4). □

In the next result we consider a six-dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\dim \mathfrak{g}_1 = 4$ and $\dim \mathfrak{g}_2 = 2$.

Proposition 4.2 *There is no* 4 ⊕ 2 *non-nilpotent Lie algebra admitting symplectic half-flat structure.*

Proof All the non-nilpotent Lie algebras of type 4 ⊕ 2 with decomposable four-dimensional one are isomorphic to one of Proposition 4.1 except the Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$. So in order to avoid the study of the same Lie algebras several times, from now on we will refer to the $4 \oplus 2$ Lie algebras as the ones that being decomposable in the form $4 \oplus 2$ cannot be decomposed as $3 \oplus 3$. The only $4 \oplus 2$ non-nilpotent Lie algebras admitting both half-flat structure [11] and symplectic form are the non-unimodular Lie algebras given in Table 5. These four possible cases are easily rejected thanks to the obstruction (i) in Proposition 2.1: it suffices to consider the 1-form $\alpha = e^5$ for $A_{4,1} \oplus \mathfrak{r}_2$, $\alpha = e^4$ for $A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$, $\alpha = e^3$ in case $A_{4,12} \oplus \mathfrak{r}_2$ and $\alpha = e^1 + e^3$ for $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$. □

In the next result we study the $5 \oplus 1$ decomposable Lie algebras, i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}$.

Proposition 4.3 *The only* $5 \oplus 1$ *non-nilpotent Lie algebras admitting symplectic half-flat structures are* $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ *and* $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ *with* $\alpha \geq 0$.

Proof As in the previous proposition, every Lie algebra of type $5 \oplus 1$ with decomposable five-dimensional part is isomorphic to one of types $3 \oplus 3$ or $4 \oplus 2$. In the list of $5 \oplus 1$ non-nilpotent Lie algebras admitting half-flat structures [11], those having also symplectic form are given in Table 6 of the Appendix. (In particular, the only $5 \oplus 1$ non-unimodular solvable Lie algebras having both half-flat structure and symplectic form are $A_{5,36} \oplus \mathbb{R}$ and $A_{5,37} \oplus \mathbb{R}$.)

In the case of $A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ with $0 < \beta < 1$, $A_{5,8}^{-1} \oplus \mathbb{R}$, $A_{5,19}^{-1,2} \oplus \mathbb{R}$, $A_{5,36} \oplus \mathbb{R}$ and $A_{5,37} \oplus \mathbb{R}$ we find that obstruction (ii) in Proposition 2.1 is satisfied because $F(\tilde{J}_\rho e_1, e_1) = 0$.

As $Z^2(A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}) = Z^2(A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}) = Z^2(A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R})$ and $Z^3(A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}) = Z^3(A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}) = Z^3(A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R})$ for any γ, the same conclusion as for $A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ is valid. Moreover, we have that $Z^2(A_{5,14}^{0} \oplus \mathbb{R}) = Z^2(A_{5,8}^{-1} \oplus \mathbb{R})$ and $Z^3(A_{5,14}^{0} \oplus \mathbb{R}) = Z^3(A_{5,8}^{-1} \oplus \mathbb{R})$. Therefore, the Lie algebras $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$, $A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$ and $A_{5,14}^{0} \oplus \mathbb{R}$ do not admit symplectic half-flat structure.

For the case $\mathfrak{g} = A_{5,15}^{-1} \oplus \mathbb{R}$ we have that an arbitrary pair $(F, \rho) \in Z^2(\mathfrak{g}) \times Z^3(\mathfrak{g})$ is given by

$$
\begin{aligned}
F &= b_1(e^{14} - e^{23}) + b_2 e^{15} + b_3 e^{24} + b_4 e^{25} + b_5 e^{35} + b_6 e^{45} + b_7 e^{56},\\
\rho &= a_1 e^{125} + a_2 e^{135} + a_3 e^{145} + a_4(e^{146} - e^{236}) + a_5 e^{156} + a_6 e^{235} + a_7 e^{245}\\
&\quad + a_8 e^{246} + a_9 e^{256} + a_{10} e^{345} + a_{11} e^{356} + a_{12} e^{456}.
\end{aligned}
$$

It follows that $F^3 \neq 0$ if and only if $b_1, b_7 \neq 0$, so we can take $b_1 = 1$. When imposing $F \wedge \rho = 0$ we have that $a_3 = a_6 - a_2 b_3$, $a_4 = a_5 = 0$, $a_9 = a_8 b_2$, $a_{11} = 0$ and $a_{12} = -a_8 b_5$. A direct calculation shows that $\tilde{J}_\rho e_1 = a_2 a_8\, e_1$, so $F(\tilde{J}_\rho e_1, e_1) = 0$. Similarly, for $A_{5,18}^{0} \oplus \mathbb{R}$ we get that $\tilde{J}_\rho e_1$ is a multiple of e_1. Thus, by Proposition 2.1 the Lie algebras $A_{5,15}^{-1} \oplus \mathbb{R}$ and $A_{5,18}^{0} \oplus \mathbb{R}$ do not admit symplectic half-flat structure.

Finally, the half-flat structures given in [11] for $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$ satisfy that $dF = 0$, so they provide examples of symplectic half-flat structures (see Table 6). □

Notice that the previous propositions imply that if $\mathfrak{g}$ is a decomposable non-unimodular solvable Lie algebra then $\mathfrak{g}$ does not admit symplectic half-flat structure.

Remark 4.4 The Lie algebra $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ is unimodular and the corresponding simply connected Lie group admits a compact quotient as it is shown in [23]. For the simply connected Lie groups corresponding to $A_{5,7}^{-1,-1,1}$ and $A_{5,17}^{\alpha,-\alpha,1}$ with $\alpha \geq 0$, conditions for the existence of lattice are given in [3, Propositions 7.2.1 and 7.2.14]. In particular, there is a lattice for the cases $A_{5,7}^{-1,-1,1}$ and $A_{5,17}^{\alpha,-\alpha,1}$ for $\alpha = 0$ and for some positive α.

5 Symplectic half-flat structures and SUSY equations of type IIA

In this section we consider as in [1,14] type II supergravity backgrounds which are warped products of the Minkowski space $\mathbb{R}^{3,1}$ and a compact manifold M of dimension 6, with non-trivial fluxes living on the internal manifold M. In order to get (at least) $N = 1$ supersymmetry, it is required the existence of (at least) a pair of globally defined and non-vanishing spinors on M satisfying the SUSY conditions. The existence of such a pair implies a reduction of the structure group of the tangent bundle to a subgroup $G \subset SO(6)$, so that different types of G structures arise.

The existence of a globally defined non-vanishing spinor η_+ on M defines a reduction to SU(3), that is, M is endowed with an SU(3) structure (F, Ψ). On the other hand, an SU(2) structure on a six-dimensional manifold M is defined by two orthogonal globally defined spinors η_+ and χ_+, which we can suppose of unit norm, or equivalently by an almost Hermitian structure (J, g), a $(1, 0)$-form α such that $\|\alpha\|^2 = 2$, a real 2-form ω and a $(2, 0)$-form Ω satisfying the conditions

$$\omega^2 = \frac{1}{2}\Omega \wedge \overline{\Omega} \neq 0, \quad \omega \wedge \Omega = 0, \quad \Omega \wedge \Omega = 0$$

and

$$i_\alpha \Omega = 0, \quad i_\alpha \omega = 0,$$

where i_α denotes the contraction by the vector field dual to α.

Notice that the SU(2) structure (α, ω, Ω) is naturally embedded in the SU(3) structure defined by η_+ as

$$F = \omega + \frac{i}{2}\alpha \wedge \overline{\alpha}, \qquad \Psi = \alpha \wedge \Omega; \tag{3}$$

and conversely, given an SU(3) structure (F, Ψ) and a $(1, 0)$-form α of norm $\sqrt{2}$ on M, then $\omega = F - \frac{i}{2}\alpha \wedge \overline{\alpha}$ and $\Omega = \frac{1}{2} i_{\overline{\alpha}} \Psi$ provide an SU(2) structure.

Now a rotation $k_{||}\eta_+ + k_\perp \chi_+$ of the two orthogonal spinors η_+ and χ_+, where $k_{||} = \cos(\phi)$ and $k_\perp = \sin(\phi)$, $\phi \in [0, \frac{\pi}{2}]$, gives rise to the family of SU(2) structures $(\alpha, \tilde{\omega}_\phi, \tilde{\Omega}_\phi)$ on M given by

$$\begin{aligned} \tilde{\omega}_\phi &= \cos(2\phi)\, \omega + \sin(2\phi)\mathrm{Re}(\Omega), \\ \tilde{\Omega}_\phi &= -\sin(2\phi)\, \omega + \cos(2\phi)\mathrm{Re}(\Omega) + i\mathrm{Im}(\Omega). \end{aligned} \tag{4}$$

Definition 5.1 [1] The SU(2) structure defined by (4) is called intermediate if $k_{||}$ and $k_\perp$ are both different from zero.

By working in the projection (eigen)basis, Andriot obtained in [1] the SUSY equations of type IIA for intermediate SU(2) structures in the following form:

$$\begin{cases} d(\mathrm{Re}(\alpha)) = 0, \\ d(\mathrm{Re}(\Omega)_\perp) = k_{||}k_\perp \mathrm{Re}(\alpha) \wedge d(\mathrm{Im}(\alpha)), \\ d(\mathrm{Im}(\Omega)) \wedge \mathrm{Re}(\alpha) = -d(\mathrm{Im}(\alpha) \wedge \mathrm{Re}(\Omega)_{||}), \end{cases} \tag{5}$$

where $\mathrm{Re}(\Omega)_\| = \frac{1}{2}((1-\cos(2\phi))\mathrm{Re}(\Omega)+\sin(2\phi)\omega)$ and $\mathrm{Re}(\Omega)_\perp = \frac{1}{2}((1+\cos(2\phi))\mathrm{Re}(\Omega) - \sin(2\phi)\omega)$.

Actually, the SUSY equations of type IIA consist of Eq. (5) together with the corresponding fluxes F_0, F_2, F_4 and H, but they can be obtained explicitly from these equations (see [1] for details).

Next we recall the precise relationship between solutions of (5) and the symplectic half-flat condition. In the following result, by a *symplectic half-flat* SU(2) *structure* (α, ω, Ω) we mean that the associated SU(3) structure given by (3) is symplectic half-flat.

Theorem 5.2 [10, Theorem 3.1] *If* $(M, \alpha, \omega, \Omega)$ *is a six-dimensional manifold endowed with an SU(2) structure such that the forms* α, $\mathrm{Re}(\Omega)_{||}$, $\mathrm{Re}(\Omega)_{\perp}$ *and* $\mathrm{Im}(\Omega)$ *satisfy Eq.* (5)*, then M admits a symplectic half-flat SU(2) structure* $(\hat{\alpha}, \hat{\omega}, \hat{\Omega})$ *with* $d(\mathrm{Re}(\hat{\alpha})) = 0$*. Conversely, if M has a symplectic half-flat SU(2) structure* $(\hat{\alpha}, \hat{\omega}, \hat{\Omega})$ *such that* $d(\mathrm{Re}(\hat{\alpha})) = 0$*, then the forms* α, $\mathrm{Re}(\Omega)_{||}$, $\mathrm{Re}(\Omega)_{\perp}$ *and* $\mathrm{Im}(\Omega)$ *defined by*

$$\frac{1}{k_{\perp}}\mathrm{Re}(\Omega)_{\perp} = \hat{\omega}, \quad \mathrm{Im}(\Omega) - i\frac{1}{k_{||}}\mathrm{Re}(\Omega)_{||} = \hat{\Omega}, \quad \mathrm{Re}(\alpha) + ik_{||}\mathrm{Im}(\alpha) = \hat{\alpha}$$

provide a solution of Eq. (5)*.*

Particular examples on compact solvmanifolds corresponding to $\mathfrak{g}_{5,1} \oplus \mathbb{R}$, $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$ are given in [10]. Other solvable Lie algebras related to the SUSY equations appear in [1,2,14]. As a consequence of Theorem 1.1 we obtain the complete list of compact solvmanifolds providing solutions to Eq. (5).

Proposition 5.3 *Any six-dimensional compact solvmanifold admitting an invariant symplectic half-flat structure also admits a solution of the SUSY equations of type IIA.*

Proof From Theorem 5.2 it is sufficient to show that the solvmanifolds corresponding to the Lie algebras listed in Theorem 1.1 admit an SU(2) structure (α, ω, Ω) with $\mathrm{Re}(\alpha)$ a closed form and such that the SU(3) structure $(F = \omega + \frac{i}{2}\alpha \wedge \overline{\alpha}, \Psi = \alpha \wedge \Omega)$ is symplectic half-flat. We give an explicit example for each case:

For $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ we consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^1 + i\,e^4$, $\omega = e^{23} + 2e^{56}$ and $\Omega = (e^2 + i\,e^3) \wedge ((e^5 - e^6) + i(e^5 + e^6))$.

For $\mathfrak{g}_{5,1} \oplus \mathbb{R}$ we consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^1 + i\,e^4$, $\omega = e^{25} - e^{36}$ and $\Omega = (e^2 + i\,e^5) \wedge (-e^3 + i\,e^6)$.

For $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ we consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^5 + i\,e^6$, $\omega = -e^{13} + e^{24}$ and $\Omega = (e^3 + i\,e^1) \wedge (e^2 + i\,e^4)$.

For $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ we consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^5 + i\,e^6$, $\omega = e^{13} + e^{24}$ and $\Omega = (e^1 + i\,e^3) \wedge (e^2 + i\,e^4)$.

For the nilpotent Lie algebra $\mathfrak{g}_{6,N3}$, since e^1 is closed, we can consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^1 + i\,e^6$, $\omega = -2e^{34} - e^{25}$ and $\Omega = (2e^4 + i\,e^3) \wedge (e^5 + i\,e^2)$.

For $\mathfrak{g}_{6,38}^{0}$, since e^6 is closed, we can consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^6 - 2i\,e^1$, $\omega = e^{34} + e^{52}$ and $\Omega = (e^3 + i\,e^4) \wedge (e^5 + i\,e^2)$.

For $\mathfrak{g}_{6,54}^{0,-1}$ we consider the SU(2) structure (α, ω, Ω) given by $\alpha = e^5 + i\,e^6$, $\omega = e^{14} + e^{23}$ and $\Omega = (e^1 + i\,e^4) \wedge (e^2 + i\,e^3)$.

Finally, for the Lie algebra $\mathfrak{g}_{6,118}^{0,-1,-1}$ we consider the structure (α, ω, Ω) given by $\alpha = e^6 + i\,e^5$, $\omega = e^{14} + e^{23}$ and $\Omega = (e^1 + i\,e^4) \wedge (e^2 + i\,e^3)$. □

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Appendix

In this appendix we summarize the results obtained in the before sections. In Table 1, we show the structure equations of all *indecomposable unimodular* solvable Lie algebras of dimension 6 having symplectic forms, obtained in [18]. Also in Table 2 we show the structure equations of *indecomposable non-unimodular* solvable Lie algebras with five-dimensional nilradical admitting symplectic forms according the proof of Proposition 3.2, while in Table 3 we consider the *indecomposable non-unimodular* solvable Lie algebras with four-dimensional nilradical having both half-flat structure and symplectic forms, but only two of those Lie algebras have symplectic half-flat structure. Moreover, we show the structure equations of all decomposable (non-nilpotent) solvable Lie algebras of dimension 6 which admit both symplectic and half-flat structures (Tables 4, 5, 6).

Table 1 Indecomposable unimodular solvable (non-nilpotent) Lie algebras admitting symplectic structures [18]

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$\mathfrak{g}_{6,3}^{0,-1}$	$\left(e^{26}, e^{36}, 0, e^{46}, -e^{56}, 0\right)$	–
$\mathfrak{g}_{6,10}^{0,0}$	$\left(e^{26}, e^{36}, 0, e^{56}, -e^{46}, 0\right)$	–
$\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$	$\left(-\frac{1}{2}e^{16} + e^{23}, -e^{26}, \frac{1}{2}e^{36}, e^{46}, 0, 0\right)$	–
$\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$	$\left(-\frac{1}{2}e^{16} + e^{23}, \frac{1}{2}e^{26}, -e^{36}, e^{46}, 0, 0\right)$	–
$\mathfrak{g}_{6,15}^{-1}$	$\left(e^{23}, e^{26}, -e^{36}, e^{26} + e^{46}, e^{36} - e^{56}, 0\right)$	–
$\mathfrak{g}_{6,18}^{-1,-1}$	$\left(e^{23}, -e^{26}, e^{36}, e^{36} + e^{46}, -e^{56}, 0\right)$	–
$\mathfrak{g}_{6,21}^{0}$	$\left(e^{23}, 0, e^{26}, e^{46}, -e^{56}, 0\right)$	–
$\mathfrak{g}_{6,36}^{0,0}$	$\left(e^{23}, 0, e^{26}, -e^{56}, e^{46}, 0\right)$	–
$\mathfrak{g}_{6,38}^{0}$	$\left(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0\right)$	$F = -2e^{16} + e^{34} - e^{25}$ $\rho = -2e^{135} - 2e^{124} + e^{236} - e^{456}$
$\mathfrak{g}_{6,54}^{0,-1}$	$\left(e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0\right)$	$F = e^{14} + e^{23} + e^{56}$ $\rho = e^{125} - e^{136} + e^{246} + e^{345}$
$\mathfrak{g}_{6,70}^{0,0}$	$\left(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0\right)$	–
$\mathfrak{g}_{6,78}$	$\left(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36} + e^{46}, e^{46}, -e^{56}, 0\right)$	–
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$\left(-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0\right)$	$F = e^{14} + e^{23} - e^{56}$ $\rho = e^{126} - e^{135} + e^{245} + e^{346}$
$\mathfrak{n}_{6,84}^{\pm 1}$	$\left(-e^{45}, -e^{15} - e^{36}, -e^{14} + e^{26} \mp e^{56}, e^{56}, -e^{46}, 0\right)$	–

Table 2 Indecomposable non-unimodular solvable Lie algebras with five-dimensional nilradical admitting both symplectic and half-flat structure [12]

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$	$\left(-\frac{1}{3}e^{16}+e^{23}, -\frac{2}{3}e^{26}, \frac{1}{3}e^{36}, e^{46}, -e^{56}, 0\right)$	$F=-2e^{16}+e^{34}+e^{52}$ $\rho=-2e^{135}-2e^{124}-e^{356}+e^{246}$
$A_{6,39}^{\frac{3}{2},-\frac{3}{2}}$	$\left(-\frac{1}{2}e^{16}+e^{45}, e^{15}+\frac{1}{2}e^{26}, \frac{3}{2}e^{36}, -\frac{3}{2}e^{46}, e^{56}, 0\right)$	–
$A_{6,39}^{1,-1}$	$\left(e^{45}, e^{15}+e^{26}, e^{36}, -e^{46}, e^{56}, 0\right)$	–
$A_{6,42}^{-1}$	$\left(e^{45}, e^{15}+e^{26}, e^{36}+e^{56}, -e^{46}, e^{56}, 0\right)$	–
$A_{6,51}^{\pm 1}$	$\left(e^{45}, e^{15}\pm e^{46}, e^{36}, 0, 0, 0\right)$	–
$A_{6,54}^{-1,-2}$	$\left(e^{16}+e^{35}, -2e^{26}+e^{45}, 2e^{36}, -e^{46}, -e^{56}, 0\right)$	–
$A_{6,54}^{\alpha,\alpha-1}$ $0<\alpha<2$	$\left(e^{16}+e^{35}, (\alpha-1)e^{26}+e^{45}, (1-\alpha)e^{36}, -e^{46}, \alpha e^{56}, 0\right)$	–
$A_{6,54}^{2,1}$	$\left(e^{16}+e^{35}, e^{26}+e^{45}, -e^{36}, -e^{46}, 2e^{56}, 0\right)$	$F=e^{31}+e^{42}+2e^{65}$ $\rho=e^{346}+e^{235}-e^{145}-2e^{126}$
$A_{6,56}^{1}$	$\left(e^{16}+e^{35}, e^{36}+e^{45}, 0, -e^{46}, e^{56}, 0\right)$	–
$A_{6,65}^{1,2}$	$\left(e^{16}+e^{35}, e^{16}+e^{26}+e^{45}, -e^{36}, e^{36}-e^{46}, 2e^{56}, 0\right)$	–
$A_{6,70}^{\alpha,\frac{\alpha}{2}}$ $\alpha\neq 0$	$\left(\frac{\alpha}{2}e^{16}-e^{26}+e^{35}, e^{16}+\frac{\alpha}{2}e^{26}+e^{45},\right.$ $\left.-\frac{\alpha}{2}e^{36}-e^{46}, e^{36}-\frac{\alpha}{2}e^{46}, \alpha e^{56}, 0\right)$	$F=e^{13}+e^{24}-\alpha e^{65}$ $\rho=-\alpha e^{126}-e^{145}+e^{235}+\alpha e^{346}$
$A_{6,71}^{-\frac{3}{2}}$	$\left(\frac{3}{2}e^{16}+e^{25}, \frac{1}{2}e^{26}+e^{35}, -\frac{1}{2}e^{36}+e^{45}, -\frac{3}{2}e^{46}, e^{56}, 0\right)$	$F=e^{41}+e^{23}+2e^{56}$ $\rho=-e^{245}+2e^{346}-2e^{126}-e^{135}$
$A_{6,76}^{-3}$	$\left(-5e^{16}+e^{25}, -2e^{26}+e^{45}, e^{24}-e^{36}, e^{46}, -3e^{56}, 0\right)$	–
$A_{6,82}^{2,5,9}$	$\left(2e^{16}+e^{24}+e^{35}, 6e^{26}, 10e^{36}, -4e^{46}, -8e^{56}, 0\right)$	–
$A_{6,94}^{-3}$	$\left(-e^{16}+e^{25}+e^{34}, -2e^{26}+e^{35}, -3e^{36}, 2e^{46}, e^{56}, 0\right)$	–
$A_{6,94}^{-\frac{5}{3}}$	$\left(\frac{1}{3}e^{16}+e^{25}+e^{34}, -\frac{2}{3}e^{26}+e^{35}, -\frac{5}{3}e^{36}, 2e^{46}, e^{56}, 0\right)$	–
$A_{6,94}^{-1}$	$\left(e^{16}+e^{25}+e^{34}, e^{35}, -e^{36}, 2e^{46}, e^{56}, 0\right)$	–

Table 3 Indecomposable non-unimodular symplectic solvable Lie algebras with four-dimensional nilradical

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$N_{6,1}^{\alpha,\beta,-\alpha,-\beta}$ $\alpha\beta\neq 0$	$\left(\alpha e^{15}+\beta e^{16}, -\alpha e^{25}-\beta e^{26}, e^{36}, e^{45}, 0, 0\right)$	–
$N_{6,1}^{\alpha,\beta,0,-1}$ $\alpha\beta\neq 0$	$\left(\alpha e^{15}+\beta e^{16}, -e^{26}, e^{36}, e^{45}, 0, 0\right)$	–
$N_{6,1}^{\alpha,\beta,-1,0}$ $\alpha\beta\neq 0$	$\left(\alpha e^{15}+\beta e^{16}, -e^{25}, e^{36}, e^{45}, 0, 0\right)$	–
$N_{6,2}^{-1,\beta,-\beta}$	$\left(-e^{15}+\beta e^{16}, e^{25}-\beta e^{26}, e^{36}, e^{35}+e^{46}, 0, 0\right)$	–
$N_{6,2}^{0,-1,\gamma}$	$\left(-e^{16}, e^{25}+\gamma e^{26}, e^{36}, e^{35}+e^{46}, 0, 0\right)$	–

Table 3 continued

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$N_{6,7}^{0,\beta,0}$ $\beta \neq 0$	$\left(-e^{26}, e^{16}, e^{35}, e^{35} + \beta e^{36} + e^{45}, 0, 0\right)$	–
$N_{6,13}^{\alpha,\beta,-\alpha,-\beta}$ $\alpha^2 + \beta^2 \neq 0$ $(\alpha, \beta) \neq (0, \pm 2)$	$\left(\alpha e^{15} + \beta e^{16}, -\alpha e^{25} - \beta e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0\right)$	–
$N_{6,13}^{0,-2,0,2}$	$\left(-2e^{16}, 2e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0\right)$	$F = e^{12} + e^{35} + e^{46}$ $\rho = e^{134} - e^{156} - e^{236} + e^{245}$
$N_{6,14}^{\alpha,\beta,0}$ $\alpha\beta \neq 0$	$\left(\alpha e^{15} + \beta e^{16}, e^{26}, -e^{45}, e^{35}, 0, 0\right)$	–
$N_{6,15}^{0,\beta,\gamma,0}$ $\beta \neq 0$	$\left(e^{15} + \gamma e^{16} - e^{26}, e^{16} + e^{25} + \gamma e^{26}, -\beta e^{45}, \beta e^{35}, 0, 0\right)$	–
$N_{6,16}^{0,0}$	$\left(e^{16}, e^{15} + e^{26}, -e^{45}, e^{35}, 0, 0\right)$	–
$N_{6,17}^{0}$	$\left(0, e^{15}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0\right)$	–
$N_{6,18}^{0,\beta,0}$ $\beta \neq 0$	$\left(e^{16} - e^{25}, e^{15} + e^{26}, -\beta e^{45}, \beta e^{35}, 0, 0\right)$	–
$N_{6,20}^{0,-1}$	$\left(-e^{56}, -e^{26}, e^{36}, e^{45}, 0, 0\right)$	–
$N_{6,22}^{\alpha,0}$ $\alpha \neq 0$	$\left(e^{15} + \alpha e^{16}, e^{26}, 0, e^{35}, 0, 0\right)$	–
$N_{6,23}^{\alpha,0}$	$\left(e^{15} - e^{26}, e^{16} + e^{25}, 0, e^{35} + \alpha e^{36}, 0, 0\right)$	–
$N_{6,26}^{0}$	$\left(-e^{56}, e^{26}, -e^{45}, e^{35}, 0, 0\right)$	–
$N_{6,28}$	$\left(-e^{24} + e^{15}, -e^{34} + e^{26}, -e^{35} + 2e^{36}, e^{45} - e^{46}, 0, 0\right)$	–
$N_{6,29}^{\alpha,\beta}$ $\alpha^2 + \beta^2 \neq 0$	$\left(-e^{23} + e^{15} + e^{16}, e^{25}, e^{36}, \alpha e^{45} + \beta e^{46}, 0, 0\right)$	–
$N_{6,30}^{\alpha}$	$\left(-e^{23} + 2e^{15}, e^{25}, e^{26} + e^{35}, \alpha e^{45} + e^{46}, 0, 0\right)$	–
$N_{6,32}^{\alpha}$	$\left(-e^{23} + e^{45} + e^{16}, e^{25} + \alpha e^{26}, (1-\alpha)e^{36} - e^{35}, e^{46}, 0, 0\right)$	–
$N_{6,33}$	$\left(-e^{23} + e^{15} + e^{16}, e^{25}, e^{36}, e^{36} + e^{46}, 0, 0\right)$	–
$N_{6,34}^{\alpha}$	$\left(-e^{23} + e^{15} + (1+\alpha)e^{16}, e^{25} + \alpha e^{26}, e^{36}, e^{35} + e^{46}, 0, 0\right)$	–
$N_{6,35}^{\alpha,\beta}$ $\alpha \neq 0$	$\left(-e^{23} + 2e^{16}, -e^{35} + e^{26}, e^{36} + e^{25}, \alpha e^{45} + \beta e^{46}, 0, 0\right)$	–
$N_{6,37}^{\alpha}$	$\left(-e^{23} + e^{45} + 2e^{16}, e^{26} - e^{35} - \alpha e^{36}, e^{25} + \alpha e^{26} + e^{36}, 2e^{46}, 0, 0\right)$	–
$N_{6,38}$	$\left(-e^{23} + e^{15} + e^{16}, e^{25}, e^{36}, -e^{56}, 0, 0\right)$	–
$N_{6,39}$	$\left(-e^{23} + 2e^{16}, -e^{35} + e^{26}, e^{25} + e^{36}, -e^{56}, 0, 0\right)$	–

Table 4 $3 \oplus 3$ decomposable (non-nilpotent) Lie algebras admitting both symplectic and half-flat structures

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$\mathfrak{e}(2) \oplus \mathfrak{e}(2)$	$\left(0, -e^{13}, e^{12}, 0, -e^{46}, e^{45}\right)$	–
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$\left(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}\right)$	$F = e^{14} + e^{23} + 2e^{56}$ $\rho = e^{125} - e^{126} - e^{135} - e^{136}$ $+ e^{245} + e^{246} + e^{345} - e^{346}$
$\mathfrak{e}(2) \oplus \mathbb{R}^3$	$\left(0, -e^{13}, e^{12}, 0, 0, 0\right)$	–
$\mathfrak{e}(1,1) \oplus \mathbb{R}^3$	$\left(0, -e^{13}, -e^{12}, 0, 0, 0\right)$	–
$\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$\left(0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45}\right)$	–
$\mathfrak{e}(2) \oplus \mathfrak{h}$	$\left(0, -e^{13}, e^{12}, 0, 0, e^{45}\right)$	–
$\mathfrak{e}(1,1) \oplus \mathfrak{h}$	$\left(0, -e^{13}, -e^{12}, 0, 0, e^{45}\right)$	–
$\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\left(0, -e^{13}, e^{12}, 0, -e^{45}, 0\right)$	–
$\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\left(0, -e^{13}, -e^{12}, 0, -e^{45}, 0\right)$	–

Table 5 $4 \oplus 2$ decomposable (non-nilpotent) Lie algebras admitting both symplectic and half-flat structures

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$A_{4,1} \oplus \mathfrak{r}_2$	$\left(e^{24}, e^{34}, 0, 0, 0, e^{56}\right)$	–
$A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$	$\left(\frac{1}{2}e^{14} + e^{23}, e^{24}, -\frac{1}{2}e^{34}, 0, 0, e^{56}\right)$	–
$A_{4,12} \oplus \mathfrak{r}_2$	$\left(e^{13} + e^{24}, -e^{14} + e^{23}, 0, 0, 0, e^{56}\right)$	–
$\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$	$\left(0, -e^{12}, 0, -e^{34}, 0, -e^{56}\right)$	–

Table 6 $5 \oplus 1$ decomposable (non-nilpotent) Lie algebras admitting both symplectic and half-flat structures

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ $0 < \beta < 1$	$\left(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45}, 0, 0\right)$	–
$A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$	$\left(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0\right)$	$F = -e^{13} + e^{24} + e^{56}$ $\rho = -e^{126} - e^{145} - e^{235} - e^{346}$
$A_{5,8}^{-1} \oplus \mathbb{R}$	$\left(e^{25}, 0, e^{35}, -e^{45}, 0, 0\right)$	–
$A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ $\gamma > 0$	$\left(e^{15}, -e^{25}, \gamma e^{45}, -\gamma e^{35}, 0, 0\right)$	–
$A_{5,14}^{0} \oplus \mathbb{R}$	$\left(e^{25}, 0, e^{45}, -e^{35}, 0, 0\right)$	–
$A_{5,15}^{-1} \oplus \mathbb{R}$	$\left(e^{15} + e^{25}, e^{25}, -e^{35} + e^{45}, -e^{45}, 0, 0\right)$	–
$A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$ $0 < \gamma < 1$	$\left(e^{25}, -e^{15}, \gamma e^{45}, -\gamma e^{35}, 0, 0\right)$	–
$A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ $\alpha \geq 0$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25},$ $-\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0)$	$F = e^{13} + e^{24} + e^{56}$ $\rho = e^{125} - e^{146} + e^{236} - e^{345}$

Table 6 continued

$\mathfrak{g}$	Str. equations	Symplectic half-flat str.
$A_{5,18}^{0} \oplus \mathbb{R}$	$(e^{25} + e^{35}, -e^{15} + e^{45}, e^{45}, -e^{35}, 0, 0)$	–
$A_{5,19}^{-1,2} \oplus \mathbb{R}$	$\left(-e^{15} + e^{23}, e^{25}, -2e^{35}, 2e^{45}, 0, 0\right)$	–
$A_{5,36} \oplus \mathbb{R}$	$\left(e^{14} + e^{23}, e^{24} - e^{25}, e^{35}, 0, 0, 0\right)$	–
$A_{5,37} \oplus \mathbb{R}$	$\left(2e^{14} + e^{23}, e^{24} + e^{35}, -e^{25} + e^{34}, 0, 0, 0\right)$	–

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